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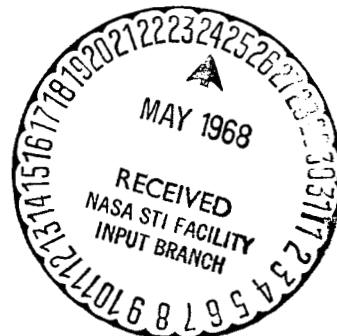
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PROBLEM OF OPTIMUM COMMANDS OF A VERTICAL TAKE-OFF AIRCRAFT

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ABSTRACT: Discussion of the control laws which will provide attainment of a prescribed altitude and acceleration to a prescribed speed in minimum time or minimum fuel consumption, for a VTOL aircraft. Some considerations of the command variables are given, involving a generalized theory of this concept and guidelines for using this theory to construct a mathematical model. The statement of optimum problems is examined for aircraft with and without thrust control, using the couple of the longitudinal command, the magnitude and direction of the thrust, the angle of attack, and the inclination of the velocity vector to the horizon as the command variables. The results are used to define the arcs of which the optimum trajectories are composed.

1. INTRODUCTION, GENERAL BACKGROUND, HYPOTHESES

In a previous note [1], an attempt was made to determine the law of variation of the magnitude and orientation of the thrust so as to insure the most economical ascent and acceleration in a minimum of time, considering the case of a VTOL aircraft with a mobile thrust axis relative to the body of the aircraft. In this paper, an attempt will be made to complete the results contained in the previous paper and to consider in addition the problem of optimum controls for the formula of the axis of fixed thrust relative to the aircraft. / 659* / 660

To begin, we shall recapitulate certain general assertions and hypotheses given in [1], which also remain valid under the conditions required by the object of the present work.

We shall consider the VTOL aircraft, whose main thrust developed by the engines provide for propulsion along the trajectory as well as for the lift, integral or partial, under specific regimes. The distribution of the components of

*Numbers in the margin indicate pagination in the foreign text.

propulsion and lift is essentially achieved (I) by the rotation of the thrust axis relative to the aircraft in the plane of symmetry of the latter, as in [1], or (II) by a change of the longitudinal position of the craft, relative to which the resultant thrust axis is fixed (for example, parallel to the zero lift axis of the aircraft). The main thrust does not participate in the attitude control of the aircraft, since it is assumed that the thrust vector always passes through its center of gravity. In either of these cases, the control of the longitudinal position, the only one to be considered, is achieved either by the elevator when the aerodynamic velocity is sufficient to make it efficient, or, for regimes of quasi-zero velocity, by suitably arranged compressed air jet controls, or finally by a combination of these two principles of control.

For reasons of convenience and simplicity of exposition, we shall assume that the mass of the aircraft and the specific mass of the air are constant; that the thrust is independent of the flight regime (in particular, of the velocity), and depends only on the intake of gases required by the respective organ of control. It is also assumed that, for the remaining fixed parameters, the aerodynamic forces and moments are proportional to the square of the velocity in all the flight regimes where their action is taken into account, that the coefficients of the aerodynamic forces depend only on the angle of attack, namely, the lift coefficient linearly is that of the drag parabolically (except in cases of maneuvers of aircraft with a mobile thrust axis at large angles of attack, positive or negative, where the drag coefficients will be considered constant) and that the coefficient of the pitching moment also depends, with the exception of the angle of attack, on the angular velocity of pitching and of α .

The optimum will be sought in the entire space without imposing any restriction on the horizontal displacement.

As in [1], use will be made of mathematical methods resulting from the maximum principle of L. S. Pontryagin [2]. In order to be able to apply this theoretical method, it will be assumed that the controls lack inertia, i. e., that the control variables change from a certain value to any other value instantaneously. The class of adopted control variables will therefore be the class of continuous functions in parts, the curve representing each control variable as a function of time admitting of a finite number of points of discontinuity of the first kind. At these points, the system of equations of optimum controlled motion will undergo variations of structure which will be expressed by discontinuities in the time derivative of the state variables. It is very useful in these effective calculations to determine these junction points of the various arcs of the optimum trajectory. To this end, one is led to construct functions which will be termed commutation functions, by means of which the respective junction points will be determined. Below, the commutation laws will be presented in the form of synthetic tables.

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The article begins with considerations of control variables. A theoretical generalization of the notion and an adequate classification are proposed, and the

manner in which the variables should be used in constructing a mathematical model is indicated (Section 2). The following sections deal with the problem of the stated law of optimum guidance (ascent and acceleration in minimum time or with minimum consumption). Section 3 considers the case of VTOL aircraft with orientable thrust by dropping the hypothesis of the constant longitudinal position used in [1]. For the case of aircraft with a fixed thrust axis, Section 4 establishes the general equations of the longitudinal motions in the adopted hypotheses, and an attempt is made to find the laws of optimum variation of the values of the longitudinal control couple and of the thrust (Section 5), of the angle of attack (Section 6) and of the angle of the velocity vector to the horizon (Section 7).

2. CONTROL VARIABLES

A first step in the conception of a theoretical model of a complex process is usually the choice of variables (parameters) considered essential for the effects studied. In the given case of controlled flight, a part of the variables chosen - the state variables - characterize the mechanical motion in the aircraft, whereas the remaining ones, the control variables, characterize the law of guidance. In the following treatment, the mathematical models will be conceived without considering the process, whatever may be their nature (mechanical, electrical, physico-physiological, etc.), which take place within the chain of guidance, from the control movement of the pilot or servo to the physical quantity represented by the control variable, by replacing them with hypotheses which define the functional relation between these quantities of, let us say, entrance and exit. In this manner, the state and control variables will be generally quantities of the same nature (geometric, cinematic, or dynamic). However, in contrast to the state variables, determined by the system of equations considered, the control variables will figure in the equations as external relations, functions of time, imposed to the system. It is to be noted, however, that the optimum problems do not necessarily / 662 require the consideration of such open systems. On the contrary, it would be convenient from the standpoint of programming of the organ of calculation of the autopilot, for example, to obtain results in a synthesized form, $u = u(t, x)$, where u is the control vector and x is the state vector. However, the consideration of closed systems in optimization problems involves great mathematical difficulties.

It is assumed that in a certain area defined by physical considerations, the values of the control variables can be imposed arbitrarily at every instant, and their optimum variation laws are sought, for example, in one of the two meanings defined above. Thus, the mathematical model of a controlled system contains, in addition to independent variable t , two types of variables: the control variables which can be imposed arbitrarily, and the state variables which result uniquely defined if the following quantities are given: the equations conforming to the hypotheses made; a system of initial conditions enclosed by the physical problem

being treated; the law of variations of the control variables with time. However, since as was mentioned above, the two categories of variables represent quantities of the same physical nature, there is always a certain degree of arbitrariness in the choice of control variables of the totality of variables which characterize the dynamics of the system as a whole. The choice will be influenced by formal considerations such as the simplification of the mathematical problem to be solved, of the calculations to be carried out, etc. Before beginning an analysis of this aspect, which is of great importance to the objective of this paper, a classification of the control variables will be proposed below.

As already mentioned, in the mathematical models used in this work, no consideration is given to the processes which take place in the control chain providing for the link between the control movement executed by the pilot or the servo and the effect of this movement on the value of the respective control variable having the same physical nature as one or several state variables. If the processes neglected by the replacement of the movement of control as such by a "control variable" in the indicated sense is independent of the remaining variables of the system (whether they are variables of state or control), the respective control variable will be named directly. As direct control variables, one can cite the angle of inclination of the elevator, the position of the gas inlet vane or the thrust of the engine itself, assuming that it is independent of the flight velocity and altitude, etc.

As control variables, one can also use certain kinematic parameters whose value is determined by the control movement of the pilot or servo by modifying the dynamic table of the object being controlled as a whole. In other words, in this case, the chain which links the control movements of the pilot or servo to the physical quantity represented by the control variable chosen contains elements which depend on at least one of the state variables or control variables of the equations of motion. We shall say in this case that we are dealing with an indirect control variable. Obviously, the condition of being able to impose the value of each control variable arbitrarily, generally is not compatible with the complete system of equations based on the physical hypotheses adopted ahead of time. One is thus led to additional hypotheses which, while restricting the generality of the problem, would eliminate the incompatibilities that occur. Thus, for example, in order to be able to use the angle of attack or the angle of longitudinal position as control variables as well, it is necessary to assume (I) that the external forces, i. e., those tangential and normal to the trajectory, acting on the aircraft, are independent of the respective angular velocities (α or q^*) and (II) that the range of variation of the kinematic parameters on which the couple of longitudinal control depends ($\alpha, M, q, \dot{\alpha}$, etc.) permits the latter to assume a sufficiently large value at every instant. Obviously, this last hypothesis is not satisfied a priori. Hence, an additional hypothesis is required. An example

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*For symbols which are not defined in the text, see the list of principal symbols given at the end of the article.

of an additional hypothesis whose necessity is imposed by the use of the inclination of the trajectory as a control variable will be given in Section 7.

3. AIRCRAFT WITH ORIENTATION OF THE CONTROLLABLE THRUST

Reference [1] considered the case of main thrust whose angular position relative to the body of the aircraft can be varied by means of a control. The rotation of the thrust axis takes place in the plane of symmetry of the aircraft around the center of gravity of the latter. φ was used to denote the angle formed by the thrust vector with the axis of zero lift of the aircraft, whose positive direction is considered to be that of the forward direction (from the trailing edge to the leading edge). A study was made of the problem of ascent to a given altitude and acceleration until a given horizontal velocity is reached (in particular, for example, the altitude of the cruising velocity) with a minimum total consumption or in a minimum of time. An attempt was made to find the optimization of the law of variation of the thrust (replaced by the adimensional quantity $\tau = T/mg$, $m = \text{const}$, $g = \text{const}$, which will be called the specific thrust) and the variation of angle φ , assuming that the longitudinal position is constant. In the treatment below, the simplifying hypothesis $\theta = \text{const}$ is dropped, and instead of a third-order system of differential equations (system (2.1) of [1]) we consider the following system of fifth order:

$$\begin{aligned} \frac{dV}{dt} &= -(1-\eta)k_1V^2 - \eta(k_2 + k_3\alpha^2)V^2 - g\sin\gamma + g\tau\cos(\varphi + \alpha) \equiv U, \\ \frac{dh}{dt} &= V\sin\gamma, \\ \frac{d\gamma}{dt} &= (1-\eta)k_4V + \eta k_5V\alpha - \frac{g\cos\gamma}{V} + \frac{g\tau\sin(\varphi + \alpha)}{V} \equiv \Gamma, \\ \frac{d\alpha}{dt} &= q - \frac{d\gamma}{dt} \equiv \mathcal{A}, \\ \frac{dq}{dt} &= \mu_1(V, \gamma, \alpha, q) + \eta k_6V\tau\sin(\varphi + \alpha) + \mu_2(V, \alpha, \delta, \delta_1) \equiv Q \end{aligned} \quad (3.1)$$

whence

$$\eta = \begin{cases} 1 & \text{for } -\alpha^{(1)} \leq \alpha \leq \alpha^{(2)}, \\ 0 & \text{for the remainder} \end{cases}$$

$$k_1 = \frac{\rho S C_{D_1}}{2}, k_2 = \frac{\rho S C_{D_2}}{2}, k_3 = \frac{\rho S C_{L_{\alpha}}}{2\pi\lambda}, k_4 = \frac{\rho S C_{L_{\alpha}}}{2},$$

$$k_5 = \frac{\rho S C_{L_{\alpha}}}{2}, k_6 = -\frac{\rho g S C_{m_{\dot{\alpha}}}}{2B}.$$

To within a constant factor (B), the sum of the first two terms of the second member of the last equation represents the aerodynamic pitching moment, and μ_2 , to within the same constant factor, represents the longitudinal control couple. As control variables, we select the quantities τ , φ and μ_2 , subjected to the following constraints:

$$0 \leq \tau \leq \tau_{\max}, 0 \leq \varphi \leq \frac{\pi}{2} + \epsilon, \mu_2^{(1)}(V, \alpha) \leq \mu_2 \leq \mu_2^{(2)}(V, \alpha), \quad (3.2)$$

where ϵ is a given constant angle (usually less than $\pi/4$).

Let t_1 (free t_1) be the time which the aircraft takes to reach a given altitude h_1 at a minimum consumption and with a horizontal velocity which is also given, V_1 .

Assuming that the consumption per unit time is a function of only the total thrust developed by the engines, respectively of the specific thrust τ , let $f(\tau)$ be this function; the problem of minimum consumption amounts to the minimization of the functional

$$J = \int_0^{t_1} f(\tau) dt. \quad (3.3)$$

It is assumed that for small values of argument τ the function $f(\tau)$ is small, and the derivative $f'(\tau)$ is very large (in particular, it is even assumed that $f(0) = 0$ and $\lim_{\tau \rightarrow 0} f'(\tau) = \infty$). It is also assumed that the second derivative $f''(\tau)$ retains its sign, at least in the accepted range of τ . For example, the function f may be of the following form: $f(\tau) = \kappa \tau^\beta$, where the constants κ and β satisfy the inequalities $\kappa > 0$ and $0 < \beta < 1$.

In conformity with Pontryagin's maximum principle, the optimum variation laws of the selected control variables, if they exist, should be sought among those / 665 which insure that at each instant the function

$$H = -f(\tau) + \psi_1 U + \psi_2 V \sin \gamma + \psi_3 \Gamma + \psi_4 \alpha + \psi_5 Q \quad (3.4)$$

reaches its maximum value relative to these control variables; the auxiliary variables ψ satisfy the adjoint system

$$\begin{aligned} \frac{d\psi_1}{dt} &= -\frac{\partial H}{\partial V}, \quad \frac{d\psi_2}{dt} = -\frac{\partial H}{\partial h}, \quad \frac{d\psi_3}{dt} = -\frac{\partial H}{\partial \gamma}, \quad \frac{d\psi_4}{dt} = \\ &= -\frac{\partial H}{\partial \alpha}, \quad \frac{d\psi_5}{dt} = -\frac{\partial H}{\partial q}. \end{aligned} \quad (3.5)$$

Obviously, H will reach its maximum relative to the control variables (the state and auxiliary variables being considered fixed) at the same time as the function

$$\begin{aligned} \mathcal{H} = & -f(\tau) + \psi_1 g \tau \cos(\varphi + \alpha) + \frac{\psi_3 - \psi_4}{V} g \tau \sin(\varphi + \alpha) + \\ & + \psi_5 [\eta k_0 V \tau \sin(\varphi + \alpha) + \mu_2] \end{aligned} \quad (3.6)$$

obtained from H by replacing the symbols U , Γ , α and Q by their expressions [see Eq. (3.1)] and then omitting the terms which do not contain the control variables. Making the substitution

$$\psi_1 g = \sigma \sin \psi, \quad \frac{g(\psi_3 - \psi_4)}{V} - \psi_5 \eta k_0 V = \sigma \cos \psi, \quad (3.7)$$

the expression of $\mathcal{H}$ becomes

$$\mathcal{H} = -f(\tau) + \sigma \tau \sin(\psi - \varphi - \alpha) + \psi_5 \mu_2. \quad (3.8)$$

It can be readily seen that the law of variation of μ_2 , which at each instant insures the maximum value of the function $\mathcal{H}$, is, to some extent, a bang-bang law, i. e., at each instant μ_2 will have either the maximum value or the minimum value compatible with the values at the respective instant of the state variables V and α : $\mu_2^{(v)}(V, \alpha)$ or $\mu_2^{(u)}(V, \alpha)$ depending on whether the sign of ψ , which serves as a commutation function, will be positive or negative (see Table 1).

TABLE 1. Law of Optimum Variation of Longitudinal Control

Commutation condition	μ_0 optimum
$\psi_0 > 0$	$\mu_0^{(0)}(V, \alpha)$
$\psi_0 < 0$	$\mu_0^{(1)}(V, \alpha)$

As in the simplified case treated in [1], in the case considered here, the law of variation of the specific thrust τ , subjected to the constraint $\tau_{\min} \leq \tau \leq \tau_{\max}$, with $\tau_{\min}$ and $\tau_{\max}$ constant, which insures the minimizations of the functional 3.3, is also an ordinary bang-bang; the optimum thrust will be maximum or minimum depending on whether the value of a certain commutation function will be higher or lower than a critical quantity equal to $f(\tau_{\max})/\tau_{\max}$. In the simplified case of [1], this commutation function depended explicitly only on the auxiliary variables ψ_1 and ψ_2 . When the hypothesis $0 = \text{const}$ is dropped, the commutation function Σ becomes more complicated; it will have the expression $\Sigma = \sigma \sin(\psi - \varphi - \alpha)$ and will hence depend [see (3.5)], in addition to the auxiliary variables ψ_1, ψ_2, ψ_3 and ψ_4 , on the state variables V and α and on the control variable φ . That is why the optimum law of the thrust is related to that of angle φ . Since σ is positive, it follows from (3.8) that the value of φ which makes the function Σ maximum depends on the function $\psi - \alpha$, in conformity with the first two columns of Table 2. The optimum law of the value of the thrust which results is in conformity with the values of the commutation function Σ obtained by taking the first two columns of the table into account.

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TABLE 2. Optimum Laws of the Magnitude and Orientation of the Total Thrust*

Commutation conditions for optimum	φ Optimum	Commutation conditions for optimum τ^{**}	τ Optimum
$2k\pi < \psi - \alpha < \frac{\pi}{2} + 2k\pi$	0	$\Sigma < \Sigma_{cr}$ $\Sigma \geq \Sigma_{cr}$	0 $\tau_{\max}$
$\frac{\pi}{2} + 2k\pi < \psi - \alpha < \pi + 2k\pi$	$\psi - \alpha - \frac{\pi}{2} + 2k\pi$	$\Sigma \geq \Sigma_{cr}$ $\Sigma < \Sigma_{cr}$	$\tau_{\max}$ 0
$\pi + 2k\pi < \psi - \alpha < \frac{3\pi}{2} + 2k\pi$	$\frac{\pi}{2} + 2k\pi$	$\Sigma < \Sigma_{cr}$ $\Sigma \geq \Sigma_{cr}$	0 $\tau_{\max}$
$\frac{3\pi}{2} + 2k\pi < \psi - \alpha < 4k\pi$	Indifferent***	$\Sigma < 0 < \Sigma_{cr}$	0

* $k = 0, 1, 2, 3, \dots$

** $\Sigma = \sigma \sin(\psi - \varphi - \alpha)$. The critical value of this commutation function $\Sigma_{cr} = f(\tau_{\max})/\tau_{\max}$.

***In the last line of the table, the optimum orientation of the thrust which results is indifferent a posteriori. Indeed, since $\Sigma < 0 < \Sigma_{cr}$, it follows that optimum is zero.

The control variables τ , φ and μ_2 which insure a minimum duration of the maneuver considered, if such values exist, should make the function

$$\mathcal{H}^* = \sigma \tau \sin(\psi - \varphi - \alpha) + \psi_5 \mu_2. \quad (3.9)$$

maximum. As far as the control variables φ and μ_2 are concerned, the conditions of Table 1 and of the first two columns of Table 2 retain their validity for the problem of the time minimum as well. As for the variable τ a bang-bang optimum law would result for it from (3.9): the specific thrust would be τ_{max} or 0 depending on whether the commutation function Σ or, in other words, $\sin(\psi - \varphi_{opt} - \alpha)$, is positive or negative. The function $\sin(\psi - \varphi_{opt} - \alpha)$ is positive in the first three lines of Table 2 and negative in the last. It now seems intuitively evident that in order to insure the fastest ascent and acceleration, it is necessary that the thrust force be permanently a maximum. Therefore, it is quite improbable that a situation such as defined by the last line of the table and implying a zero thrust would arise. Therefore, in order to simplify the calculations without risking an appreciable error, in the problem of the time minimum one could take into account the first three lines of the table only, and neglect the control variable τ , by considering it to be constant and equal to τ_{max} .

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4. THRUST AXIS FIXED RELATIVE TO THE AIRCRAFT. GENERAL EQUATIONS

Using the adopted hypotheses and assuming in addition that the direction of the resultant thrust is permanently parallel to the axis of zero lift of the aircraft with noninclined control surfaces, the equations of the controlled motion in the plane of symmetry of the aircraft can be reduced to the form*

$$\frac{dV}{dt} = -(k_1 + k_2 \alpha^2) V^2 - g \sin \gamma + g \tau \cos \alpha = \mathcal{U}_1,$$

$$\frac{dh}{dt} = V \sin \gamma,$$

*For the meaning of the symbols k_1 , k_2 , k_3 , k_4 , see the definition of the notation used in system (3.1) (p. 5). As in system (3.1), in the last equation of system (4.1), the sum of the first two terms of the right-hand side represents the aerodynamic pitching moment, and μ_2 represents the longitudinal control couple, both to within a constant factor (the factor B).

$$\frac{d\gamma}{dt} = k_3 V \alpha - \frac{g \cos \gamma}{V} + \frac{g \tau \sin \alpha}{V} \equiv \Gamma_1, \quad (4.1)$$

$$\frac{d\alpha}{dt} = q - \frac{d\gamma}{dt} \equiv \alpha_1,$$

$$\frac{dq}{dt} = \mu_1(V, \gamma, \alpha, q) + k_3 V \tau \sin \alpha + \mu_2(V, \alpha, \delta_e, \delta_e) \equiv Q_1.$$

The control variables will be selected among the following quantities: specific function τ ; angle of attack of the aircraft α ; angle of inclination of the trajectory γ ; the ratio μ_2 of the longitudinal control couple to the longitudinal moment of inertia B subjected to the following constraints:

$$\begin{aligned} 0 < \tau < \tau_{\max}, \quad \tau_{\max} = \text{const}, \quad -\alpha^{(1)} < \alpha < \alpha^{(2)}, \\ 0 < \gamma < \frac{\pi}{2} \text{ ou } -\frac{\pi}{2} < \gamma < \frac{\pi}{2}, \quad \mu_1^{(1)}(V, \alpha) < \mu_1 < \mu_1^{(2)}(V, \alpha). \end{aligned} \quad (4.2)$$

5. ASCENT AND ACCELERATION WITH MINIMUM CONSUMPTION

The following problem will be considered: given a sufficient number of initial and final conditions (relative to the state variables), it is necessary to minimize the functional J , [see (3.3)], with the aid of the control of the thrust and of the longitudinal moment. The respective control functions τ and μ_2 will satisfy the conditions expressed by the constraints (4.2).

The optimum functions $\tau(t)$ and $\mu_2(t)$ in the sense specified above, if such functions exist, make the function

$$\begin{aligned} \mathcal{H}_1(\tau, \mu_2) = & -f(\tau) + g\tau \left[\psi_1 \cos \alpha + \frac{1}{V}(\psi_3 - \psi_4 + \right. \\ & \left. k_7 \psi_5 V^2) \sin \alpha \right] + \psi_6 \mu_2 \end{aligned} \quad (5.1)$$

a maximum, where $k_7 = \frac{k_6}{g} = -\frac{\rho S C_{m \dot{\alpha}}}{2B}$. Making the substitution

$$g\tau\psi_1 = \sigma \sin \psi \cdot \frac{g\tau}{V} (-\psi_3 + \psi_4 - k_1 \psi_5 V^2) = \sigma \cos \psi \quad (5.2)$$

the expression of $\mathcal{H}_1$ becomes

$$\mathcal{H}_1(\tau, \mu_2) = -f(\tau) + \tau \sigma \sin(\psi - \alpha) + \psi_5 \mu_2. \quad (5.3)$$

It is readily seen that for the optimization of μ_2 we obtain an identical result to that found for the case of thrust with controllable orientation, and consequently Table 1 remains valid. Table 1 also turns out to be valid in the case of optimization of the longitudinal control couple for the problem of minimum time.

A result analogous to the case of mobile thrust is also obtained with respect to the control of the thrust. Indeed, the optimum law of function $\tau(t)$, in the case of thrust with a fixed direction relative to the aircraft, will also be a bang-bang type having as commutation functions $\Sigma_1 = \sigma \sin(\psi - \alpha)$ (obtained from Σ by setting $\varphi_i \equiv 0$) with the same critical value: $\Sigma_{1cr} = f(\tau_{max})/\tau_{max}$, if the function $f(\tau)$ enjoys the properties stated in the preceding section. We shall recall that for the functions $f(\tau)$ having these properties, the function of τ

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$$\mathcal{H}_2(\tau) = -f(\tau) + \Sigma_1 \tau$$

can be represented by a family of curves dependent on the parameter Σ_1 , whose variation is indicated in Fig. 1.

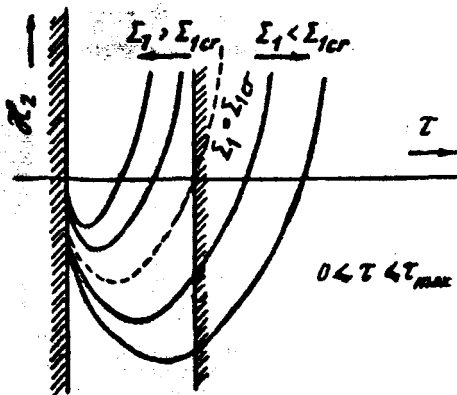


Figure 1.

The critical value of parameter Σ_1 corresponds to the curve which intersects the abscissa axis at point $\tau = \tau_{max}$, being therefore defined by the equation

$$0 = -f(\tau_{max}) + \Sigma_1 \tau_{max}.$$

Indeed, considering the constraint (4.2) relative to τ , for $\Sigma_1 > \Sigma_{1cr}$, $\mathcal{H}_2(\tau)$ reaches its highest value at the intersection of the curve $\mathcal{H}_2 = \mathcal{H}_2(\tau)$ with the straight line $\tau = \tau_{max}$, and consequently, τ optimum = τ_{max} , whereas for $\Sigma_1 < \Sigma_{1cr}$ the maximum

value of $\mathcal{H}_2(\tau)$ in the range $0 \leq \tau \leq \tau_{max}$ is the one corresponding to the origin, and hence τ optimum = 0. The result is summarized in Table 3.

TABLE 3. Function $\tau(t)$ Which Provides for the Most Economical Ascent and Acceleration (thrust parallel to the axis of zero lift of the aircraft)

Commutation condition	τ optimum
$\Sigma > f(\tau_{max})/\tau_{max}$	τ_{max}
$\Sigma_1 < f(\tau_{max})/\tau_{max}$	0

6. MINIMUM DURATION OF ASCENT AND ACCELERATION BY CONTROL OF THE ANGLE OF ATTACK

The direct control variables δ_e and δ_r (inclination of the elevator and parameter of control of the gas vane) are replaced by the angle of attack of the aircraft α - an indirect-control variable, by making the additional hypotheses discussed in Section 2 (the aerodynamic forces independent of $\dot{\alpha}$ and the values of the longitudinal control couple sufficiently large to insure with enough precision the realization of the hypothesis that $\alpha(t)$ is a continuous function in parts in the range $-\alpha^{(1)} < \alpha < \alpha^{(2)}$).

Since α is considered to be a control variable (i.e., $\alpha(t)$ as an external link imposed to the system), equation (4) relative to $d\alpha/dt$ is eliminated from (4.1), and the system formed which describes the motion of the center of gravity of the aircraft, is separated by the first three equations, from the last equation, representing the rotation about the transverse axis originating from the center of gravity. Obviously, the systems of equations modeling the motion of the center of gravity will be considered.

It is assumed that angles $\alpha^{(1)}$ and $\alpha^{(2)}$ are small enough so that the terms in α'' can be neglected if $n \geq 3$. The system to be considered will then be

$$\begin{aligned} \frac{dV}{dt} &= -(k_1 + k_2 \alpha^2) V^2 - g \sin \gamma + g \tau \left(1 - \frac{\alpha^2}{2}\right) \equiv U_1, \\ \frac{dh}{dt} &= V \sin \gamma, \\ \frac{d\gamma}{dt} &= k_3 V \alpha - \frac{g \cos \gamma}{V} + \frac{g \tau \alpha}{V} \equiv U_2. \end{aligned} \quad (6.1)$$

In conformity with the maximum principle, for optimum $\alpha(t)$, the function

$$H_2(\alpha) = \psi_1 U_2 + \psi_2 V \sin \gamma + \psi_3 \Gamma_2$$

with ψ_i satisfying the adjoint system

$$\frac{d\psi_1}{dt} = -\frac{\partial H_2}{\partial V}, \quad \frac{d\psi_2}{dt} = -\frac{\partial H_2}{\partial h}, \quad \frac{d\psi_3}{dt} = -\frac{\partial H_2}{\partial \gamma}$$

reaches its maximum value at every instant, or, in other words, the optimum $\alpha(t)$ makes the function

$$\mathcal{L}_2(\alpha) = -\psi_1 \alpha^2 \left(k_2 V^2 + \frac{g^2}{2} \right) + \psi_3 \alpha \left(k_2 V + \frac{g^2}{V} \right) \quad (6.2)$$

a maximum. Let

$$\alpha(V, \psi_1, \psi_3) = \frac{\psi_3 \left(k_2 V + \frac{g^2}{V} \right)}{2\psi_1 \left(k_2 V^2 + \frac{g^2}{2} \right)}.$$

It is then easy to see that the optimum trajectory will be made up of arcs with maximum angles of attack and other arcs with minimum angles of attack, and finally, other arcs with variable angles of attack equal at each point to $\alpha(V, \psi_1, \psi_3)$. The junction conditions are given in Table 4.

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7. MINIMUM DURATION OF THE ASCENT AND ACCELERATION BY CONTROL OF THE INCLINATION OF THE TRAJECTORY

In order to solve the optimum problem completely by using the indicated procedure, it is necessary to determine the solution of the system formed by the equation of controlled motion and by the respective adjoint equations which will satisfy the given physical conditions, in the form of initial and final values

TABLE 4. Optimum Variation of the Angle of Attack Corresponding to the Minimum Duration of the Ascent and Acceleration

Commutation conditions			α optimum
ψ_1	ψ_2	$x(V, \psi_1, \psi_2)$	
≥ 0	> 0	$> \alpha^{(2)}$	$\alpha^{(2)}$
		$\leq \alpha^{(2)}$	x
≥ 0	≤ 0	$\leq -\alpha^{(1)}$	$-\alpha^{(1)}$
		$\geq -\alpha^{(1)}$	x
≤ 0	≥ 0	Indifferently*	$\alpha^{(2)}$
≤ 0	≤ 0	$\geq \frac{1}{2}(\alpha^{(2)} - \alpha^{(1)})$	$-\alpha^{(1)}$
		$\leq \frac{1}{2}(\alpha^{(2)} - \alpha^{(1)})$	$\alpha^{(2)}$

*since, ordinary $\alpha^{(2)} > \alpha^{(1)}$ and a fortiori $\alpha^{(2)} > \alpha^{(1)} - 2x$.

and transversality conditions, etc. During the integration, the above system undergoes modifications as a function of the solution in conformity with the established commutation conditions. The construction of the algorithm and the programming of such a problem can be very cumbersome. In addition, the achievement of the required precision in the result implies the use of a trial-and-error technique with a sufficiently large number of iterations to be carried out. * ~~All this makes such a great effort undesirable if the object is merely to obtain some qualitative information, or to feel one's way, so to speak.~~ To this end, use may be made of qualitative methods which can serve to obtain the desired approximate information without recourse to integration of this system of equations. Such a method, which will be presented below, can be used to determine almost immediately the nature of the possible arcs of the optimum trajectory of the center of gravity. The method consists essentially in selecting the control variable. Thus, if the angle of the slope of the trajectory is considered as the control variable, the desired information is obtained directly from the maximum principle. It should be noted, however, that the validity of the result will be limited by the additional hypothesis required by the use of the indirect control variable γ . Indeed, by substituting the indirect control variable t , for the direct control variables δ_1 and δ_2 , of the indirect control variable γ , one eliminates from the system of equations of the controlled motion (4.1) those which determine the course of γ if the law of variation of the direct controls is given. However,

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*Some aspects of this question will be discussed in the Appendix.

this removal could cause discrepancies if an additional hypothesis justifying it were not adopted. In the case under consideration, by selecting γ as a control variable, one is led to the removal of equations (3) and (5) [relative to the derivatives of γ and q] as well as equation (4) [relative to q] containing the variable q from system (4.1). But, since α also figures in the first equation of the system (the one relative to $\dot{V}$), the removal of the three equations mentioned should be offset by imposing - as an additional hypothesis - a relation between α , γ , V , etc., justified by physical considerations. Thus, in order to eliminate the state variable α , the latter is assumed to satisfy at each instant the relation

$$\alpha = \frac{g \cos \gamma}{k_s V^2 + g \tau}, \quad k_s = \frac{\rho S C_{L\alpha}}{2}, \quad (7.1)$$

which corresponds to the following physical hypotheses: the centripetal acceleration along the optimum trajectory is negligible, and the angle of incidence is sufficiently small, so that the sine can be replaced by its argument. Incidentally, these hypotheses are not too restrictive at all, since they are currently being used in flight optimization problems.

Taking (7.1) into account, the system (4.1) is reduced to the form

$$\begin{aligned} \frac{dV}{dt} &= - \left[k_s + \frac{k_s g^2 \cos^2 \gamma}{(k_s V^2 + g \tau)^2} \right] V^2 - g \sin \gamma + g \tau \left[1 - \frac{g^2 \cos^2 \gamma}{2(k_s V^2 + g \tau)^2} \right], \\ \frac{dh}{dt} &= V \sin \gamma. \end{aligned} \quad (7.2)$$

Angle γ is to be determined so that the function

$$\mathcal{H}_2 = a \sin^2 \gamma + b \sin \gamma \quad (7.3)$$

where

$$a = \psi_2 \frac{g^2 \left(k_s V^2 + \frac{g \tau}{2} \right)}{(k_s V^2 + g \tau)^2}, \quad b = \psi_2 V - \psi_1 g$$

be a maximum at each instant. The possible resulting situations are shown by the diagrams $\mathcal{K}_2 = \mathcal{K}_2(\sin \gamma)$ [Fig. 2 for $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ and Fig. 3 for $0 < \gamma < \frac{\pi}{2}$] and are summarized in Table 5.

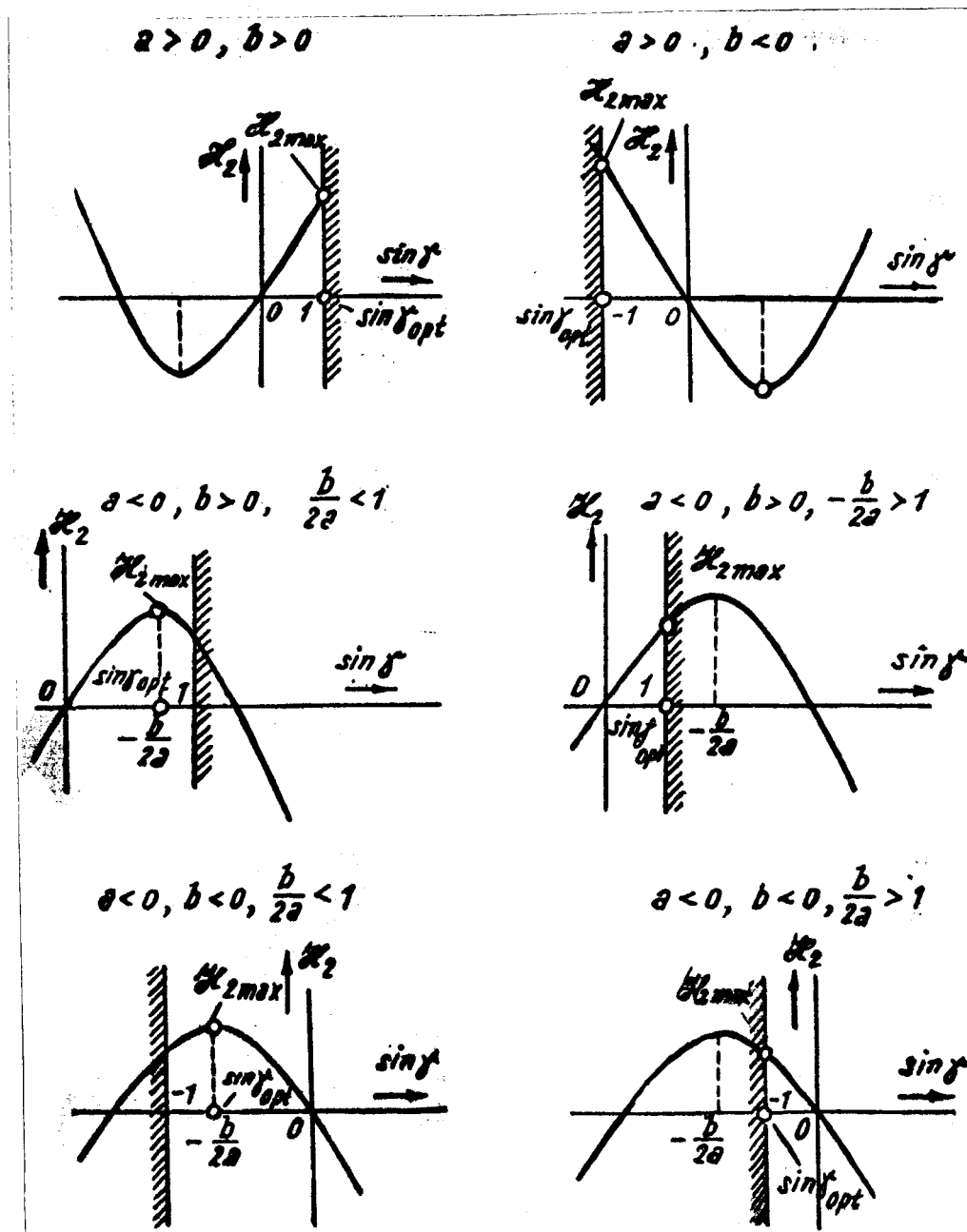


Figure 2.

It follows that the optimum trajectory of the center of gravity of the aircraft is formed by three kinds of arcs: vertical ascent, level flight (or vertical dive if γ minimum = $-\frac{\pi}{2}$) and flight with a variable slope (γ being the function of V and ψ).

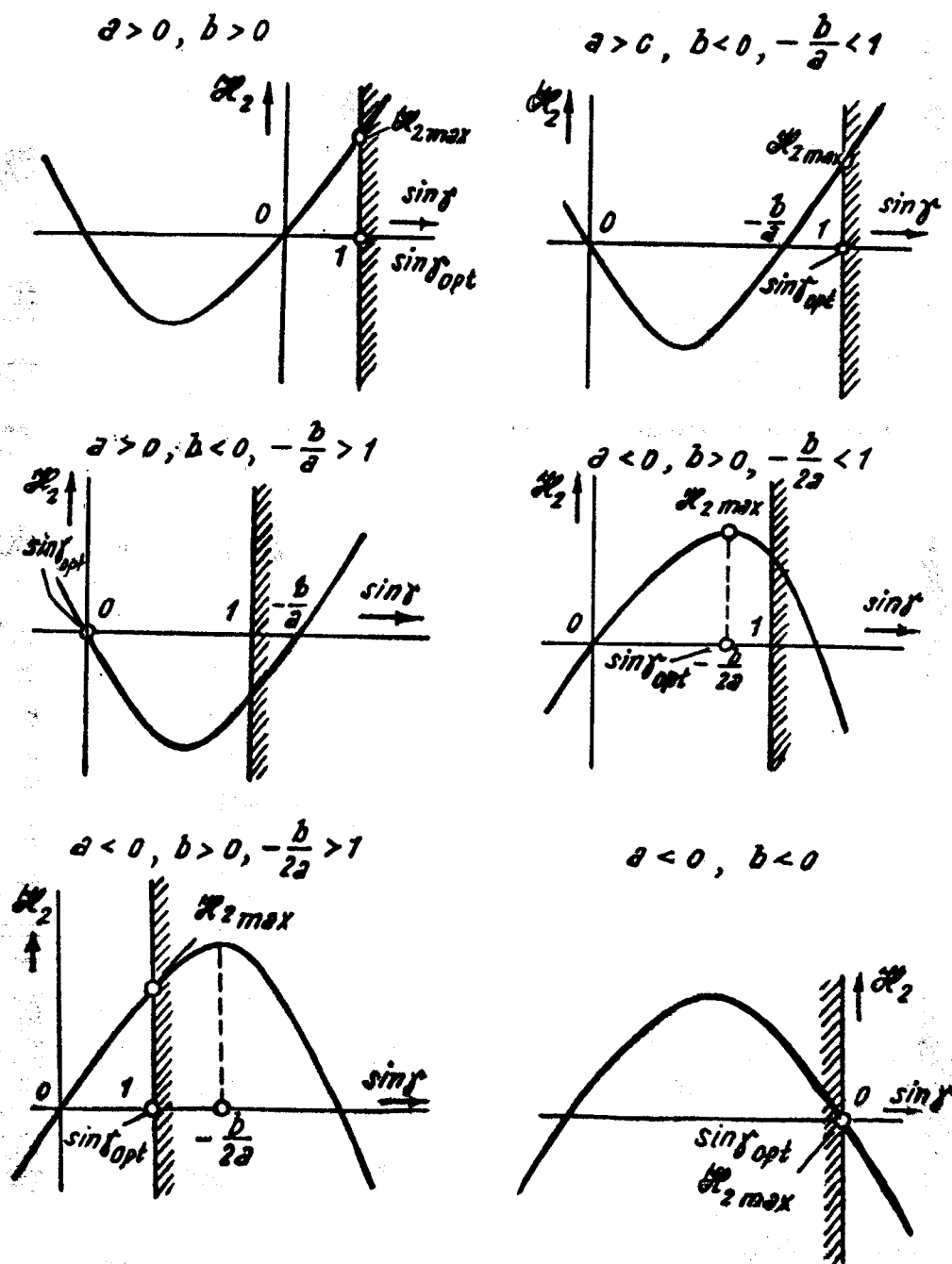


Figure 3.

TABLE 5. Optimum Variation of the Inclination of the Trajectory
Relative to the Horizon Corresponding to the Minimum
Duration of the Ascent and Acceleration

Above the range of the control variable γ	Commutation conditions			γ optimum
	a	b	$\frac{b}{2a}$	
$-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$	> 0	> 0	Indifferently	$\frac{\pi}{2}$
	> 0	< 0	Indifferently	$-\frac{\pi}{2}$
	< 0	> 0	> -1	$\arcsin\left(-\frac{b}{2a}\right)$
			< -1	$\frac{\pi}{2}$
	< 0	< 0	< 1	$\arcsin\left(-\frac{b}{2a}\right)$
			> 1	$-\frac{\pi}{2}$
$0 < \gamma < \frac{\pi}{2}$	> 0	> 0	Indifferently	$\frac{\pi}{2}$
	> 0	< 0	$> -\frac{1}{2}$	$\frac{\pi}{2}$
			$< -\frac{1}{2}$	0
	< 0	> 0	> -1	$\arcsin\left(-\frac{b}{2a}\right)$
			< -1	$\frac{\pi}{2}$
	< 0	< 0	Indifferently	0

APPENDIX

ASPECTS OF CALCULATION

The subject of the following discussion is the problem of improving the convergence of the successive approximations implied by the method which has just been exposed. In particular, a method will be presented which will make it possible to control the direction of the iteration method and to increase the precision of the numerical operations of the calculation. For the sake of a specific discussion, reference will be made to system (4.1) by taking as the initial and final configurations those indicated in Table 6.

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TABLE 6. Conditions at Extremities

t	V	h	γ	α	q
t_0	V_0^*	h_0^*	$\gamma_0 = \frac{\pi}{2}$	$\alpha_0 = 0$	$q_0 = 0$
Free	V_1	h_1	$\gamma_1 = 0$	α_1	Free

*Because of the expression of γ_1 (4.1), V_0 is given different from zero (nevertheless very small). In other words, the initial moment considered is not the precise moment of lift-off from the ground but at a later instant closer to it. Consequently, h_0 will also be different from zero and small.

In order to solve, for example, the optimum problem considered in Section 5, it is necessary to integrate the system formed by equations (4.1) and the adjoint equations corresponding to them for five given initial conditions (those of Table 6), and for five free initial conditions (the starting values of auxiliary variables ψ_i , $i = 1, 2, \dots, 5$). The algorithm established by the numerical integration obviously also contains the commutation relations, indicated for the example selected in Tables 1 and 3, according to which the system of equations varies with the solution.

The only technique possible for a bilocal problem such as just stated is the technique of trial and error. For the determination of the six free parameters; duration of the integration (of the final instant t_1) and of the five starting values of the auxiliary variables, we have six relations at our disposal, namely: four given final conditions concerning the state variables V , h , γ and α , given in Table 6, a transversality relation corresponding to the free final value of the state variable q , in the given case $\psi_5(t_1) = 0$ and finally, a relation derived from the maximum principle. In order to define this relation, the function of x is designated by $M(x)$: $\sup H_1(x, u)$, where the column vectors x and u represent the variables of the system (state and auxiliary variables) and the control variables respectively; in

the given case

$$x = \begin{pmatrix} V \\ h \\ \gamma \\ \alpha \\ q \\ \psi_1 \\ \vdots \\ \psi_r \end{pmatrix}, \quad \mu = [\mu_1]$$

H_1 is a function defined for the example considered by the formula

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$$H_1 = -f(\tau) + \psi_1 U_1 + \psi_2 V \sin \gamma + \psi_3 \Gamma_1 + \psi_4 \mathcal{A}_1 + \psi_5 Q_1,$$

with f having the properties enumerated in Section 3, and $U_1, \Gamma_1, \mathcal{A}_1, Q_1$, as for system (4.1). Finally, U represents the adopted range of the control variables [see (4.2)]. According to the maximum principle, all along the optimum trajectory the time function $\mathcal{M}(x(t))$ is constant and equal to zero. Consequently, the last relation mentioned above will be $\mathcal{M}(x(t_0)) = 0$, or $\mathcal{M}(x(t_1)) = 0$.

In order to summarize what has just been said, the algorithm will contain, for example, six given relations concerning the initial conditions (corresponding to the given instant t_0), namely:

$$\begin{aligned} V(t_0) - V_0 &= 0, \\ h(t_0) - h_0 &= 0, \\ \gamma(t_0) - \frac{\pi}{2} &= 0, \\ \alpha(t_0) &= 0, \\ q(t_0) &= 0, \\ \mathcal{M}(x(t_0)) &= 0, \end{aligned}$$

of which, in the particular case under consideration, five are identically verified by remaining a single relation:

$$\mathcal{M}(x(t_0)) = 0 \quad (\text{A. 1})$$

and five relations are given for the final conditions (which correspond to the free final date t_1):

$$\begin{aligned} F_1(x(t_1)) &\equiv V(t_1) - V_1 = 0, \\ F_2(x(t_1)) &\equiv h(t_1) - h_1 = 0, \\ F_3(x(t_1)) &\equiv \gamma(t_1) = 0, \\ F_4(x(t_1)) &\equiv \alpha(t_1) - \alpha_1 = 0, \\ F_5(x(t_1)) &\equiv \psi_5(t_1) = 0. \end{aligned} \quad (\text{A. 2})$$

The solution of the problem implies the determination of the effect of the variation of the free initial values (of the auxiliary variables) on the final conditions, which, after a number (not too large) of successive approximations, should satisfy relations (A. 2). Because of the extreme sensitivity characterizing this dependence and the precarious precision of the numerical derivation operation, / 678 several authors* have developed methods which eliminate the inconvenience of the repeated integration of the system implying the numerical derivation. The following treatment will apply the method proposed by A. H. Jazwinski in [3], for which relatively simple schemes of calculation result for the problems considered in the present article.

The following vectorial notations will be introduced:

$$X = \begin{pmatrix} X_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ X_{10} \end{pmatrix} \equiv \begin{pmatrix} V_1 \\ V \sin \gamma \\ \Gamma_1 \\ \mathcal{A}_1 \\ Q_1 \\ -\frac{\partial H_1}{\partial \psi_1} \\ \vdots \\ -\frac{\partial H_1}{\partial \psi_5} \end{pmatrix}, \quad F(x(t_1)) = \begin{bmatrix} F_1(x(t_1)) \\ \vdots \\ \vdots \\ F_5(x(t_1)) \end{bmatrix} \equiv \begin{bmatrix} V(t_1) - V_1 \\ h(t_1) - h_1 \\ \gamma(t_1) \\ \alpha(t_1) - \alpha_1 \\ \psi_5(t_1) \end{bmatrix}$$

* See [2] and [3] and [4] cited therein.

The system to be considered will then be

$$\frac{dx}{dt} = X(t, x) \quad (\text{A. 3})$$

and the final conditions (A. 2) will become

$$D(x(t_1)) = 0. \quad (\text{A. 4})$$

$x(t_0)$, and in particular the starting values of the five auxiliary variables ψ_i (the initial values of the state variables being given) are chosen so as to satisfy relation (A. 1). Let $x^{(n)}(t)$ be the solution corresponding to the initial values given for the state variables and chosen in the manner which has just been indicated for the auxiliary variables. The numerical integration is carried out up to an instant $t_1^{(n)}$ chosen according to any criterion. For example, the integration is carried out up to an instant where one of the state variables (V or h) reaches its final given value for the first time (V_1 or, respectively, h_1). In general, the solution $x^{(n)}(t)$ does not verify the final conditions (A. 2) as a whole.

A system with variations relative to the solution of $x^{(n)}(t)$ is then established:

$$\frac{dy}{dt} = \frac{\partial X}{\partial x}(t, x^{(n)}) y \quad (\text{A. 5})$$

and the adjoint system corresponding to (A. 5)

$$\frac{d\lambda}{dt} = -\lambda \frac{\partial X}{\partial x}(t, x^{(n)}), \quad (\text{A. 6})$$

where λ is a line vector with ten components: $(\lambda_1, \dots, \lambda_{10})$,

From the general properties of the adjoint system it follows that

$$\lambda(t_1^{(n)}) y(t_1^{(n)}) - \lambda(t_0) y(t_0) = 0. \quad (\text{A. 7})$$

Let $x^{(2)}(t)$ be a solution corresponding to another system of initial values $x_i(t_0)$, defined on a new interval $[t_0, t_1^{(2)}]$. We shall have

$$\begin{aligned} x^{(2)}(t_1^{(2)}) - x^{(1)}(t_1^{(1)}) &= x^{(2)}(t_1^{(1)}) - x^{(1)}(t_1^{(1)}) + x^{(2)}(t_1^{(2)}) - x^{(2)}(t_1^{(1)}) = \\ &= y(t_1^{(1)}, t_0, x^{(2)}(t_0) - x^{(1)}(t_0)) + \frac{dx}{dt}(t_1^{(2)})(t_1^{(2)} - t_1^{(1)}) + \\ &\quad + O(x^{(2)}(t_0) - x^{(1)}(t_0)) + O(t_1^{(2)} - t_1^{(1)}) \end{aligned}$$

whence we obtain

$$y(t_1^{(1)}) = x^{(2)}(t_1^{(2)}) - x^{(1)}(t_1^{(1)}) - \frac{dx}{dt}(t_1^{(2)})(t_1^{(2)} - t_1^{(1)}) + \dots$$

(A. 7) then becomes

$$\lambda(t_1^{(1)}) \left[x^{(2)}(t_1^{(2)}) - x^{(1)}(t_1^{(1)}) - \frac{dx}{dt}(t_1^{(2)})(t_1^{(2)} - t_1^{(1)}) + \dots \right] - \lambda(t_0)y(t_0) = 0. \quad (\text{A. 8})$$

The adjoint system (A. 6) is integrated in the opposite direction, ultimately considering $t_1^{(1)}$ as the initial instant and t_0 as the final instant for the systems of the following starting conditions (in the case of the example considered):

$$\begin{aligned} \lambda^1(t_1^{(1)}) &= \frac{\partial F_1}{\partial x} \Big|_{t_1^{(1)}} = (1, 0, 0, 0, \dots, 0), \\ \lambda^2(t_1^{(1)}) &= \frac{\partial F_2}{\partial x} \Big|_{t_1^{(1)}} = (0, 1, 0, 0, \dots, 0), \\ \lambda^3(t_1^{(1)}) &= \frac{\partial F_3}{\partial x} \Big|_{t_1^{(1)}} = (0, 0, 1, 0, \dots, 0), \\ \lambda^4(t_1^{(1)}) &= \frac{\partial F_4}{\partial x} \Big|_{t_1^{(1)}} = (0, 0, 0, 1, 0, \dots, 0), \\ \lambda^5(t_1^{(1)}) &= \frac{\partial F_5}{\partial x} \Big|_{t_1^{(1)}} = (0, 0, \dots, 0, 0, 1), \end{aligned}$$

which can be expressed, if the components of line vectors λ_j^k are expressed by λ_j^k , by the relations

$$\lambda_j^k(t_1^{(1)}) = \begin{cases} 1 & \text{pour } k = 1, 2, 3, 4, \text{ et } j = k \\ 0 & \text{pour } k = 1, 2, 3, 4, \text{ et } j \neq k \\ 1 & \text{pour } k = 5, j = 2k = 10 \\ 0 & \text{pour } k = 5, j \neq 2k = 10. \end{cases} \quad (\text{A. 9})$$

We shall now transform Eq. (A. 8). In order to take advantage of the specific features of the case treated to make it simpler, the scalar notation will be adopted. Since the first five components of the column vector $y(t_0)$ are zero (the values of the moment $t = t_0$ of the state variables being given), the vectorial Eq. (A. 8) can be replaced by the following system of scalar equations

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$$\sum_{j=0}^{10} \lambda_j^k(t_0) [x_j^{(2)}(t_0) - x_j^{(1)}(t_0)] + X_k(t_1^{(1)}) (t_1^{(2)} - t_1^{(1)}) + \dots = x_1^{(2)}(t_1^{(2)}) - x_1^{(1)}(t_1^{(1)}) \quad (\text{A. 10})$$

where $l = k$ for $k = 1, 2, 3, 4$ and $l = 2k = 10$ for $k = 5$. On the other hand,

$$F_k(x_1(t_1^{(2)})) - F_k(x_1(t_1^{(1)})) = \frac{\partial F_k(x_1(t_1^{(1)}))}{\partial x} [x_1^{(2)}(t_1^{(2)}) - x_1^{(1)}(t_1^{(1)})] + \dots$$

where, considering relations (A. 9),

$$F_k(x_1^{(2)}(t_1^{(2)})) - F_k(x_1^{(1)}(t_1^{(1)})) = x_1^{(2)}(t_1^{(2)}) - x_1^{(1)}(t_1^{(1)}) + \dots$$

with the same F defined above. When terms of higher order are neglected, Eqs. (A. 10) assume the form

$$\begin{aligned} \sum_{j=0}^{10} \lambda_j^k(t_0) [x_j^{(2)}(t_0) - x_j^{(1)}(t_0)] + X_k(t_1^{(1)}) (t_1^{(2)} - t_1^{(1)}) &= \\ &= F_k(x_1^{(2)}(t_1^{(2)})) - F_k(x_1^{(1)}(t_1^{(1)})) \\ k &= 1, \dots, 5 \end{aligned} \quad (\text{A. 11})$$

Thus, in order that the six unknowns $x_j^{(2)}(t_0)$, $j = 6, \dots, 10$ and $t_1^{(2)}$ (or the respective variations $x_j^{(2)}(t_0) - x_j^{(1)}(t_0)$, $j = 6, \dots, 10$ and $t_1^{(2)} - t_1^{(1)}$) be determined as functions of the variation of the final conditions $F_k(x^{(2)}(t_1^{(2)})) - F_k(x^{(1)}(t_1^{(1)}))$, $k = 1, \dots, 5$, only a single relation remains to be added. It will be obtained from the condition resulting from (A.1), namely, that the variations $x_j^{(2)}(t_0) - x_j^{(1)}(t_0)$, $j = 6, \dots, 10$ should satisfy the equation

$$\sum_{j=6}^{10} \frac{\partial \mathcal{M}}{\partial x_j} (x_j^{(1)}(t_0)) [x_j^{(2)}(t_0) - x_j^{(1)}(t_0)] = 0. \quad (\text{A.12})$$

Taking into account the given initial values (Table 6*) and the expression for H_2 at the starting point of the optimum trajectory, the last equation can be written as

$$[[-k_2 V t + g(\tau(t_0) - 1)] [\psi_1^{(2)}(t_0) - \psi_1^{(1)}(t_0)] + V_0 [\psi_2^{(2)}(t_0) - \psi_2^{(1)}(t_0)] + + \mu_2(t_0) [\psi_5^{(2)}(t_0) - \psi_5^{(1)}(t_0)] = 0 \quad (\text{A.13})$$

where the values at instant t_0 of the control variables τ and μ_2 result from Tables 1 and 3. It appears intuitively evident that the optimum $\tau(t_0)$, which can take only its maximum or zero value, will be equal to τ_{max} . The value of $\mu_2(t_0)$ will be determined as a function of the sign of $\psi_5^{(1)}(t_0)$. / 681

In conclusion, the variation of the final date, $t_1^{(2)} - t_1^{(1)}$, and of the initial values of the auxiliary variables, $\psi_i^{(2)}(t_0) - \psi_i^{(1)}(t_0)$, $i = 1, \dots, 5$ (or with the general notation introduced, $x_j^{(2)}(t_0) - x_j^{(1)}(t_0)$, $j = 6, \dots, 10$) as a function of the variation of the final conditions $F_k(x^{(2)}(t_1^{(2)})) - F_k(x^{(1)}(t_1^{(1)}))$, $k = 1, \dots, 5$, is determined with the aid of the system formed by the five Eqs. (A.11) and Eq. (A.12) [or (A.13)].

The problem may be considered closed if we find initial values $\psi_i(t_0)$ satisfying relation (A.1) and a duration of integration for which the solution of system (A.3) verifies condition (A.4) with the required accuracy. In order to approach this situation step by step, the second member of system (A.11) is chosen in the following manner:

*For these values, particularly for $\alpha(t_0) = 0$ and $q(t_0) = 0$, to which $\alpha(t_0) = 0$, can be added, in conformity with the hypothesis made, according to which the coefficient depends only on the angle of attack and on the angular velocity q and α , the moment of aerodynamic pitching, represented to within a constant factor by the two terms of the last Eq. (4.1), is equal to zero.

$$F_k(x^{(2)}(t_1^{(2)})) - F_k(x^{(1)}(t_1^{(1)})) = -c_k F_k(x^{(1)}(t_1^{(1)})),$$

where $0 < c_k \leq 1$. In this manner, the solution of (A.3) obtained from the initial values $y^{(2)}(t_0)$ resulting from the system (A.11) + (A.12) will approach the final conditions (A.4) better, since

$$|F_k(y^{(2)}(t_1^{(2)}))| = (1 - c_k) |F_k(y^{(1)}(t_1^{(1)}))| < |F_k(y^{(1)}(t_1^{(1)}))|.$$

In the reasoning given above, terms of higher order were neglected without any rigorous justification. Therefore, the omissions which were made should be justified a posteriori by the results of the calculations. Indeed, for values of c_k that are too high, such omissions appear to be inadmissible, and consequently, the solution corresponding to the initial values $x_j^{(2)}(t_0)$ and to the duration of integration $t_1^{(2)} - t_0$ obtained by the method described will not verify the starting relation $F_k(y^{(2)}(t_1^{(2)})) = (1 - c_k) F_k(y^{(1)}(t_1^{(1)}))$ with the required accuracy. It appears desirable to perform any number of attempts in order to choose suitable constants c_k (those which are as close as possible to 1).

The calculation operations described above will be repeated until a system of initial values $y_j^{(n)}(t_0)$ and a duration of integration are found which lead to the required accuracy with regard to verifying the final condition $F(y^{(n)}(t_1^{(n)})) = 0$. The number n of successive approximations will be smaller as it will be possible to select c_k values closer to 1 while retaining the validity of the reasoning.

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PRINCIPAL SYMBOLS EMPLOYED

t - time

V - velocity of the center of gravity

α - angle of attack

γ - angle of inclination of the velocity vector to the horizon

0 - longitudinal position

q - angular velocity of pitching

$$\dot{\alpha} = \frac{d\alpha}{dt},$$

h - flight altitude

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M - Mach number

m - mass of aircraft

B - longitudinal moment of inertia

S - reference surface

c - reference chord

λ - wing aspect ratio

φ - angle formed by the thrust vector with the zero lift axis of the aircraft

δ_e - inclination of the elevator

δ_r - parameter of control of jet vanes

C_L - lift coefficient

C_{D0} - drag coefficient for $C_L = 0$

C_{D1} - drag coefficient for high values of the angle of attack, here considered independent of α ,

C_{L1} - lift coefficient under the same conditions

C_m - coefficient of pitching moment

$$C_{L_\alpha} = \frac{\partial C_L}{\partial \alpha}, C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

T - main thrust

$\tau = T/mg, m = \text{const}, g = \text{const}$ (specific thrust)

g - gravitation constant

ρ - specific mass of air

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